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TERMINOLOGY

displacement
ground speed
head-to-tail
initial
linear combination
magnitude
norm
parallelogram rule
polar form
position vector
resultant
scalar
terminal point
triangle rule
unit vector
vector

VECTORS IN THE PLANE

BASIC VECTORS

1.01 Two-dimensional vectors
1.02 Addition of vectors
1.03 Component and polar forms of vectors
1.04 Multiplication by scalars
1.05 Unit vectors
1.06 Using components
1.07 Vector properties
1.08 Applications of vectors
Chapter summary
Chapter review

Prior learning

REPRESENTING VECTORS IN THE PLANE BY DIRECTED LINE SEGMENTS

- examine examples of vectors including displacement and velocity (ACMSM010)
- define and use the magnitude and direction of a vector (ACMSM011)
- represent a scalar multiple of a vector (ACMSM012)
- use the triangle rule to find the sum and difference of two vectors (ACMSM013)

ALGEBRA OF VECTORS IN THE PLANE

- use ordered pair notation and column vector notation to represent a vector (ACMSM014)
- define and use unit vectors and the perpendicular unit vectors *i* and *j* (ACMSM015)
- express a vector in component form using the unit vectors *i* and *j* (ACMSM016)
- examine and use addition and subtraction of vectors in component form (ACMSM017)
- define and use multiplication by a scalar of a vector in component form (ACMSM018) AC

1.01 TWO-DIMENSIONAL VECTORS

Vectors are used to model important aspects of the physical world. They are used to predict what will happen under different circumstances and to determine the actions that are required to achieve desired outcomes. Physics relies heavily on geometric vectors to model a wide range of phenomena, including forces, velocity, electromagnetic fields and applications, and so on. Engineers use vectors to work out how to construct cars, bridges, buildings and other objects in a safe and durable way.

Many quantities, such as mass, distance, volume and speed, are specified by a single magnitude. Quantities of this type are called **scalar** quantities. Other quantities, such as displacement, velocity and acceleration, need to have both magnitude and direction to complete their specification. These are known as **vector** quantities.

IMPORTANT

A **geometric vector** (or just **vector**) is a quantity with both **magnitude** (size) and **direction**. Both magnitude and direction must be stated to specify a geometric vector.

A vector is shown using a lower-case bold letter such as **a**. Vectors are sometimes shown by putting an arrow or wavy line over the top of the letter ($\vec{a}$ or $\tilde{a}$). In handwriting it can be shown by putting a wavy line under the lower-case letter, as in $\underset{\sim}{a}$.

The **magnitude** of a vector **p** is written as $|\mathbf{p}|$ or p. The magnitude is also called the **norm**.

A vector is shown as a **directed line segment**. The length of the segment represents the magnitude of the vector and an arrow is used to show its direction.

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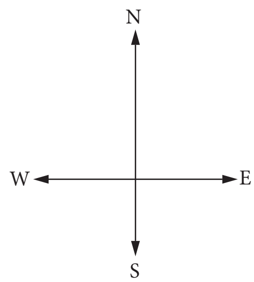
Example 1

Use directed line segments to represent the following displacements on the same set of axes.

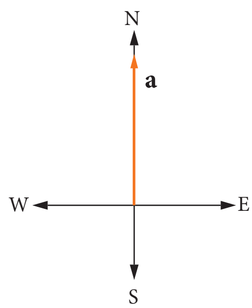
a 4 m north **b** 8 m east **c** 4 m south **d** 12 m north-west

Solution

Draw north–south and east–west lines to act as a reference when drawing the vectors.
We need to choose a suitable scale.
A scale of 1 cm = 2 m will do.



a Using the scale, **a** will be 2 cm on the diagram. Draw **a** as a 2 cm arrow in the direction of north.

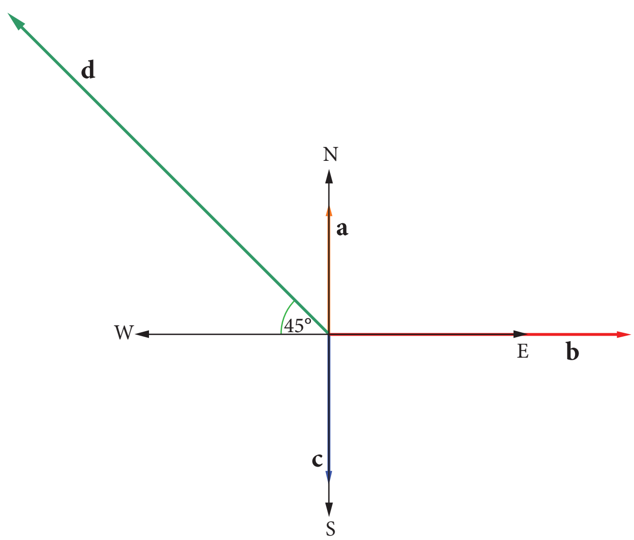


Use the same procedure for the other vectors.

b This is a 4 cm arrow in the direction of east.

c This is a 2 cm arrow in the direction of south.

d This is a 6 cm arrow in the direction of north-west.



It is more efficient and easier to show the vectors with a common origin.

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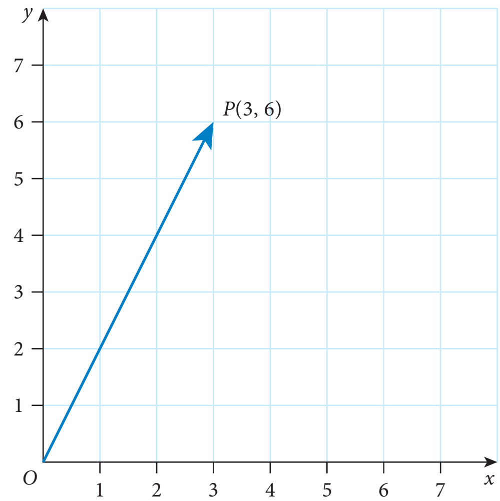
Representing vectors

A vector that refers to a fixed point or origin is called a **position vector**. The diagram on the right shows the position vector of $P(3, 6)$ with respect to the origin $O(0, 0)$ of the Cartesian plane. The position vector of P is written as **OP** or $\overrightarrow{OP}$. O is the **initial point** and P is the **terminal point** of the vector.

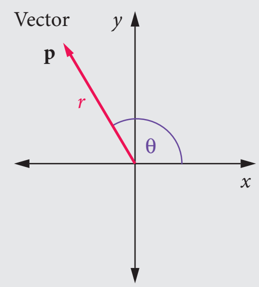
Vectors may be represented numerically by specifying the magnitude and direction.



IMPORTANT

Two-dimensional vectors are represented in polar form by an ordered pair or column vector. The first number is the magnitude of the vector and the second number gives the direction of the vector as an angle *anticlockwise* from the positive direction of the x-axis.

The two-dimensional vector **p** shown in the diagram can be written as the ordered pair (r, θ) or the column vector $\begin{bmatrix} r \\ \theta \end{bmatrix}$.



Example 2

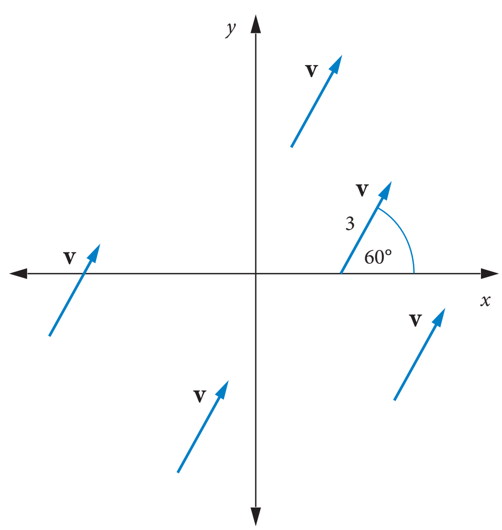
A vector **v** of magnitude 3 is at 60° to the x-axis. Show some representations of the vector.

Solution

It doesn't matter where **v** is placed. As in the diagram, the vector must have an angle of 60° to the x-axis and length 3. It has the polar representation

$\mathbf{v} = (3, 60°)$ or $\mathbf{v} = \begin{bmatrix} 3 \\ 60° \end{bmatrix}$.

Notice that the vector still has the same magnitude and direction wherever it is drawn in the plane. It is the *same vector*, although it may be drawn in different positions.



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One of the most important vector quantities in scientific applications is **displacement**. The displacement from one position to another is the *change of position*. When you look at displacement, it doesn't matter how the change has occurred. It is a geometric vector because a change of position has both magnitude and direction.

WS

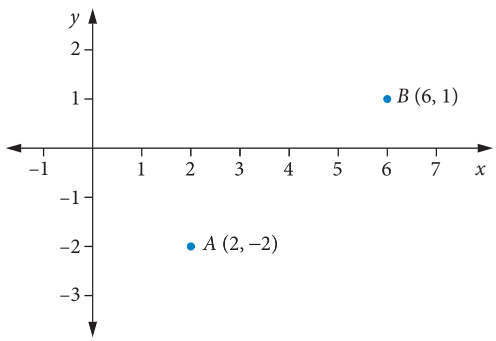
Vectors in polar form

Example 3

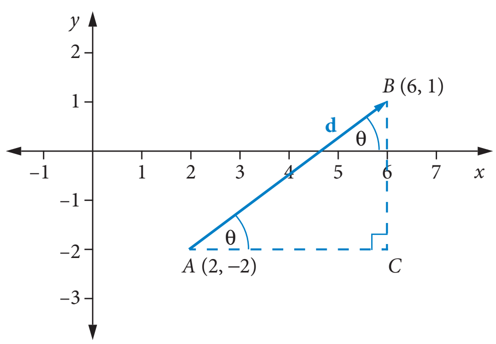
Find the displacement vector, **d**, from $A(2, -2)$ to $B(6, 1)$.

Solution

Draw a set of axes and plot $A(2, -2)$ and $B(6, 1)$.



The displacement from A to B can be shown as an arrow from A to B.
Label the arrow **d**.
You need to calculate the distance from A to B to find the magnitude of **d**, and the angle θ to find the direction.



Use the distance formula.
Substitute in the known values.
Evaluate.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6-2)^2 + (1 - -2)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Calculate θ using $\triangle ABC$ and the tan ratio.

$$\tan(\theta) = \frac{BC}{AC}$$

Substitute in the known values.

$$= \frac{3}{4}$$

Evaluate.

$$= 0.75$$

Use $\tan^{-1}$ to calculate θ.

$$\tan^{-1}(0.75) = 36.8698\ldots°$$

Round the angle and write **d** in polar form.

$$\mathbf{d} \approx (5, 36.9°)$$

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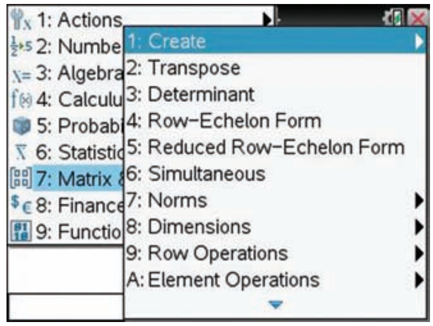
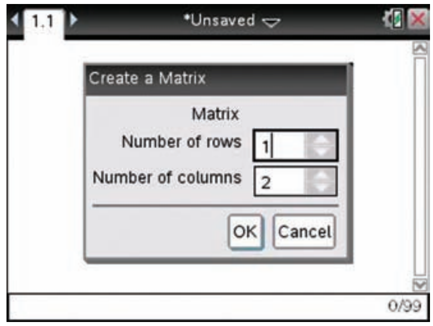
TI-Nspire CAS

You can do this on your CAS calculator by entering the coordinates as a subtraction *with the coordinates of B first*, and asking for the result in polar form.

Make sure that your calculator angle is set to degrees in your document by pressing [doc▾], 7:Settings & Status and 2: Document settings.

Then use [menu], 7:Matrix & Vector, 1:Create and 1: Matrix. Choose 1 row and 2 columns for coordinates or 2 rows and 1 column for a column vector.

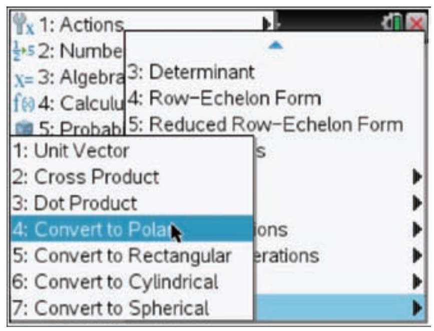


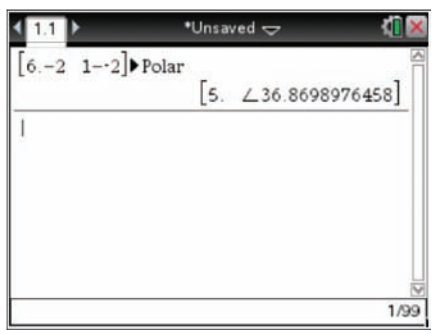


Subtract the x-coordinates in the first box, shown as [ ], and the y-coordinates in the second box.

Enter one of the coordinates with a decimal point to make the answer approximate. In the screen shown, the 6 was entered as a decimal. Make sure that you use the [(–)] button for the negative sign of –2 and the [–] button for subtraction.

Press [menu], 7:Matrix & Vector, C: Vector and 4: Convert to Polar to change the answer to polar coordinates.

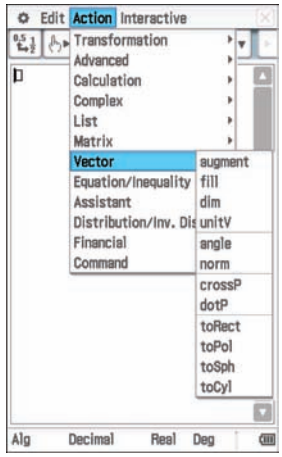




Round the angle and write **d** in polar form. **d** ≈ (5, 36.9°)

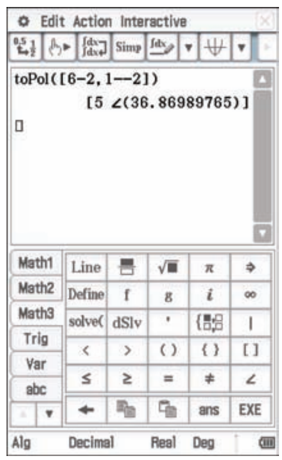
ClassPad

Tap Main $\sqrt{\alpha}$ and clear the screen by tapping **Edit** and then **Clear All** and **OK**.
Make sure the screen is set to degrees (**Deg**) and **Decimal** at the bottom.
Tap the **Action** menu at the top, then tap **Vector** and then **toPol**. (Convert to Polar.)

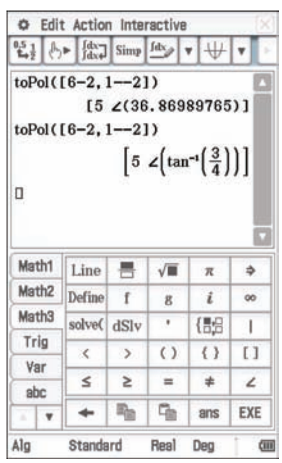


Vectors are entered using square brackets (press Keyboard then tap Math3 to see a button for [ ]) with a comma , between the elements.

Enter the vector of the difference of points, subtracting A from B ($B - A$), first the x and then the y coordinates, separated by a comma. Press EXE.



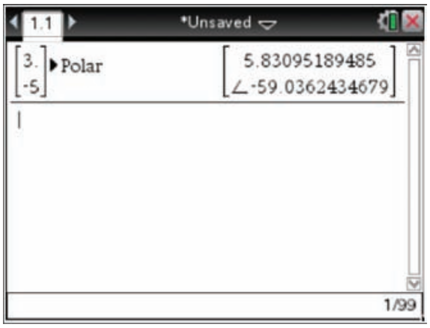
An exact answer can also be found by setting the calculator to **Standard**.



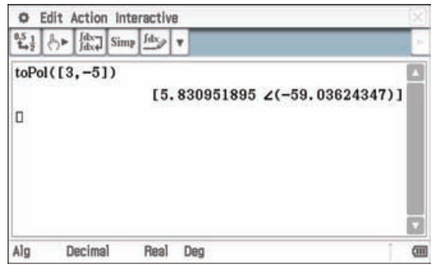
Round the angle and write **d** in polar form. **d** $\approx (5, 36.9°)$

You can use your CAS calculator to find the polar form of a position vector by entering the vector as in Example **3** and asking for the polar form. For example, the position vector for the point (3, –5) would be done as follows.

TI-Nspire CAS



ClassPad

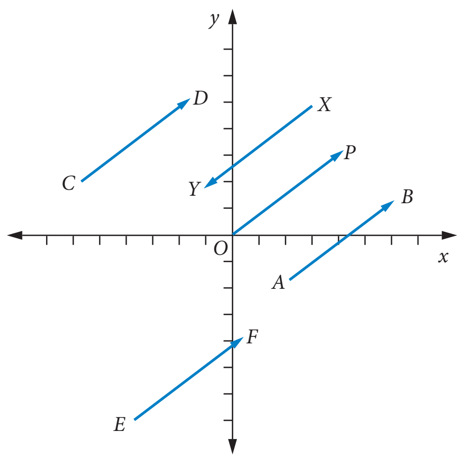


The position vector is approximately (5.8, –59.0°), or (5.8, 301.0°) with a positive angle.

Displacement vectors

Since a displacement vector may show a change from one position to another position, it is also symbolised by writing the first and second positions in order. The displacement from A to B is shown typographically as **AB**, or $\overline{AB}$ or $\overrightarrow{AB}$, and in handwriting as $\underset{\sim}{AB}$, or $\overrightarrow{AB}$.

Even though a displacement from A to B can be written using A and B, it is still the same as another vector of the same magnitude and direction in another position.



In the diagram on the right, **AB**, **CD**, **EF** and **OP** are all equal vectors. We could write

$$\mathbf{AB} = \mathbf{CD} = \mathbf{EF} = \mathbf{OP}$$

Because **XY** has the same magnitude as **OP**, but is in the opposite direction, we write

$$\mathbf{XY} = -\mathbf{OP} = -\mathbf{AB} = -\mathbf{CD} = -\mathbf{EF}$$

EXERCISE 1.01 Two-dimensional vectors

Concepts and techniques

1 Which of the following is not a vector quantity?
A displacement B speed C acceleration D force E velocity

2 What is the magnitude of the position vector of $P(5, 0)$?
A 0 B 1 C 3 D 2.5 E 5

3 What is the direction of the position vector of $Q(0, -3)$?

A $-180°$ B $0°$ C $90°$ D $180°$ E $270°$

4 A vessel is travelling south-west of a fixed point at 5 km h^{-1}. Which of the following polar forms best describes the velocity of the vessel?

A $(5, 45°)$ B $(5, 225°)$ C $(5 \text{ km h}^{-1}, 270°)$ D $(5, 135°)$ E $(5, 215°)$

5 Example 1 Use directed line segments to represent the following displacements on the same set of axes.

a 5 km west b 7 km north c 10 km south-east d 8 m south-west

6 Sketch the position vectors of each of the following points on the same set of axes.

a $(4, 7)$ b $(-3, -5)$ c $(4, -4)$ d $(-2, 6)$

7 Example 2 A vector **h** has a magnitude of 6 and is in the direction 30° south of west. Show at least five different representations of **h**.

8 Calculate the norm of the position vector for each of the following points in exact form.

a $(3, 4)$ b $(-5, 12)$ c $(24, -7)$ d $(-1, -1)$

e $(-2, 7)$ f $(3, -6)$ g $(a, 4)$ h $(6, b)$

9 Calculate the direction (in degrees correct to 1 decimal place) of the position vectors of the points in question **8** for parts **a** to **f**.

10 Sketch each of the following vectors.

a $(4, 120°)$ b $(5, 210°)$ c $(3, -40°)$ d $(5, 300°)$ e $(4, 45°)$

11 Express each of the position vectors below in polar form.

a $(3, -4)$ b $(-5, 5)$ c $(-7, -10)$ d $(4, 12)$ e $(-10, 16)$

12 Find the polar form of the position vectors for each of the following points, correct to one decimal place.

a $(8, 15)$ b $(12, 5)$ c $(-9, -12)$ d $(-7, -12)$ e $(8, -8)$

13 Example 3 Find the displacement vector (in polar form) for each of the following movements.

a $A(3, 4)$ to $B(5, 11)$ b $R(-2, 1)$ to $B(-7, -2)$ c $M(-3, -8)$ to $B(-9, 1)$

Reasoning and communication

14 A vector **u** has a magnitude of 8 and is in the direction 60° north of west.

a Describe the vector $-$**u** in words.

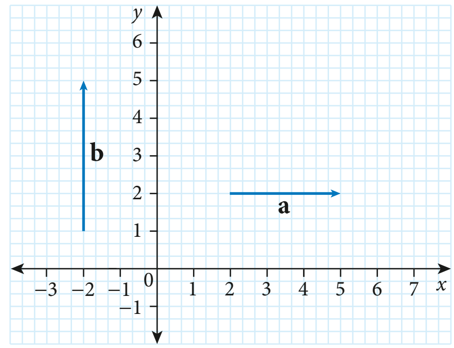
b Sketch **u** and $-$**u** on the same set of axes so that their initial points coincide.

c Sketch **u** and $-$**u** on a different set of axes so that the initial point of **u** coincides with the terminal point of $-$**u**.

d If **u** is a displacement vector, describe the displacement vector you get by doing **u** and then $-$**u**.

1.02 ADDITION OF VECTORS

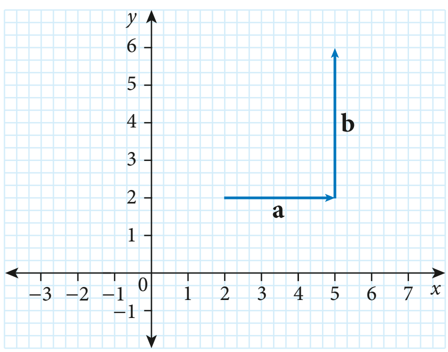
You saw in the previous section that a vector can be represented in any position on the plane. This means that we can slide or translate a vector without changing its value because its magnitude and direction have been unaltered. This is useful when we want to find the outcome of adding vectors.

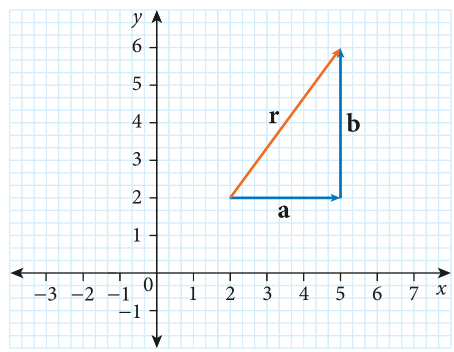


Consider the situation where there are two displacements, **a** and **b**, as shown on the right.

To find the displacement that is the same as **a** followed by **b**, slide **b** across so that its tail is on the head (tip) of **a**.

Now, the overall result of adding displacement **a** to **b** is the vector from the head of **a** to the tail of **b**. This is shown on the diagram as **r**.

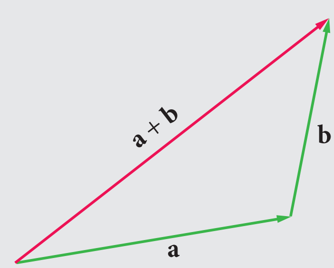




IMPORTANT

Triangle rule for vector addition

If two vectors **a** and **b** are placed head-to-tail to form two sides of a triangle, as shown in this diagram, then **a** + **b** is defined by the triangle of vectors. The sum **a** + **b** is also called the **resultant** of **a** and **b**.



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Triangle addition of vectors

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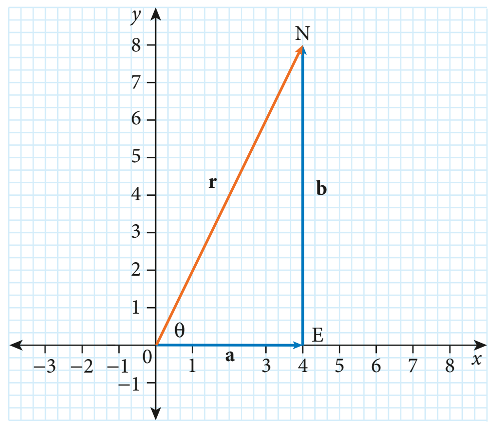
This is also called the **head-to-tail** (or **tip-to-tail**) method of vector addition.

Example 4

A boat travels due east for 4 km and then due north for 8 km. Calculate the resultant displacement of the boat.

Solution

Sketch a diagram showing the displacements **a** and **b**.
Mark in the resultant displacement, **r**.
Mark in θ as the angle made between **r** and the positive direction of the *x*-axis.



Calculate the magnitude of **r**.

$$|\mathbf{r}| = \sqrt{4^2 + 8^2}$$
$$= 4\sqrt{5}$$

Calculate the direction of **r**.

$$\theta = \tan^{-1}\left(\frac{8}{4}\right)$$
$$\approx 63.4°$$

Change to a bearing by finding the angle clockwise from north (*y*-axis).

Bearing ≈ 026.6°

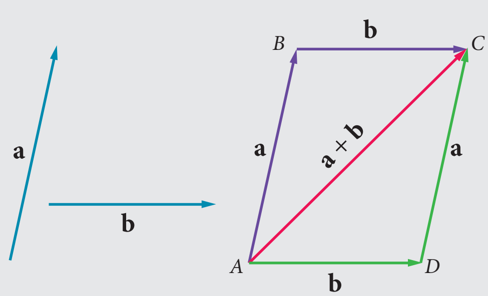
Convert the direction of **r** to a bearing and state the result with magnitude and direction.

The resultant displacement is $4\sqrt{5}$ km at a bearing of about 026.6° (or N 26.6° E).

IMPORTANT

Parallelogram rule for vector addition

Consider vectors **a** and **b** as shown on the right. If we form the *parallelogram ABCD* using **a** and **b** as the sides, the diagonal *AC* is the resultant of **a** + **b**.



Parallelogram rule for vector addition

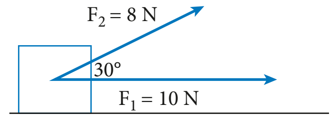
The parallelogram rule demonstrates that vector addition is commutative, as **a** + **b** = **b** + **a**.

Basic trigonometry is often used in the calculation of resultant vectors.

Example 5

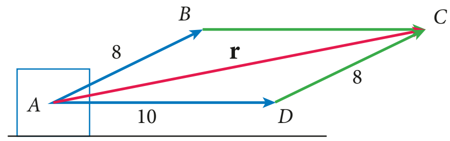
Two forces given in newtons (N) are applied to a block as shown in the diagram on the right.

Calculate the resultant force on the block.



Solution

Complete the parallelogram of vectors and label the vertices A, B, C and D.
Draw in the diagonal of the parallelogram and label it **r** (the resultant vector).



Use the properties of a parallelogram to determine the internal angles.

$\angle BAD = 30°$ (Given)
$\angle BCD = \angle BAD = 30°$ (Opposite angles of a parallelogram)
$\angle ADC = 150°$ (Co-interior angles on parallel lines)
$\angle ADC = \angle ABD = 150°$ (Opposite angles of a parallelogram)

Use the cosine rule in $\angle ADC$ to find the magnitude of **r** ($|\mathbf{r}| = r$).

$AC^2 = AD^2 + DC^2 - 2\ AD \times DC \cos(\angle ADC)$

Substitute in the known values and evaluate.

$$r^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \cos(150°)$$
$$= 302.56\ldots$$

Take the square root of both sides.

$r \approx 17.4$

Use the sine rule in $\angle ADC$ to find $\angle CAD$.

$$\frac{AC}{\sin(\angle ADC)} = \frac{CD}{\sin(\angle CAD)}$$

Substitute in the known values.

$$\frac{17.4}{\sin(150°)} = \frac{8}{\sin(\angle CAD)}$$

Rearrange and evaluate.

$$\sin(\angle CAD) = \frac{8 \times \sin(150°)}{17.4}$$
$$= 0.2299\ldots$$

Calculate $\angle CAD$.

$$\angle CAD = \sin^{-1}(0.2299\ldots)$$
$$\approx 13.3°$$

State the result.

The resultant force on the block is about 17.4 N acting at 13.3° above the horizontal.

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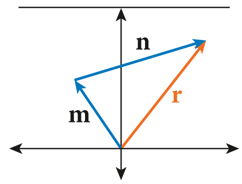
Example 6

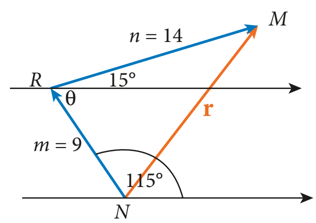
Vectors **m** and **n** are (9, 115°) and (14, 15°). Find the resultant **r** = **m** + **n**.

Solution

Position **m** with its tail at the origin.
Position **n** with its tail on the head of **m**.
Draw in the resultant vector, **r**.



Find the angle between **m** and **n**.
$\theta = 65°$ (Co-interior with 115°)
$\angle NRM = 65° + 15° = 80°$



Calculate the magnitude of **r** using the cosine rule.

$$r^2 = m^2 + n^2 - 2nm\cos(R)$$
$$= 14^2 + 9^2 - 2 \times 14 \times 9 \times \cos(80°)$$
$$= 233.2406\ldots$$

Find the square root.

$$r = \sqrt{233.2406\ldots}$$
$$= 15.2722...$$

Find $\angle RNM$ using the sine rule.

$$\frac{\sin(N)}{n} = \frac{\sin(R)}{r}$$

Use the exact value on your calculator.

$$\frac{\sin(N)}{14} = \frac{\sin(80°)}{15.2722\ldots}$$

Rearrange.

$$\sin(N) = \frac{14\sin(80°)}{15.2722\ldots}$$

Calculate.

$$= 0.9027\ldots$$

Calculate $\angle RNM$.

$$N = \sin^{-1}(0.9027\ldots)$$
$$= 64.5246\ldots$$

Find the direction of **r**.

$$\text{Direction of } \boldsymbol{r} = 115° - 64.5246\ldots°$$
$$\approx 50.5°$$

State the result.

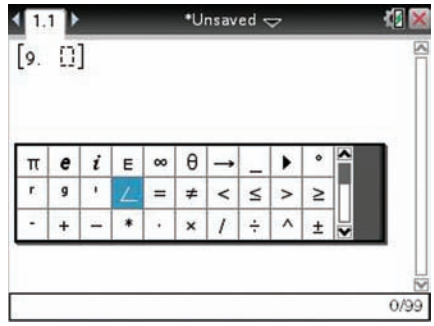
$$\mathbf{r} \approx (15.3, 50.5°)$$

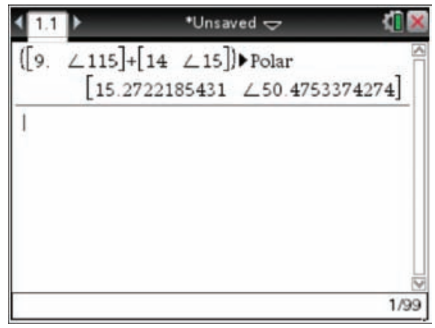
TI-Nspire CAS

Use menu, 7:Matrix & Vector, 1:Create and 1: Matrix for each vector.

The angle symbol is necessary. It is in the symbols menu. You get this by pressing ctrl and 📖.

Make sure that you ask for the answer in polar form using menu, 7:Matrix & Vector, C: Vector and 4: Convert to Polar.





Round and state the result.

r ≈ (15.3, 50.5°)

ClassPad

Tap Main $\sqrt{\alpha}$ and clear the screen by tapping **Edit** and then **Clear All** and **OK**.

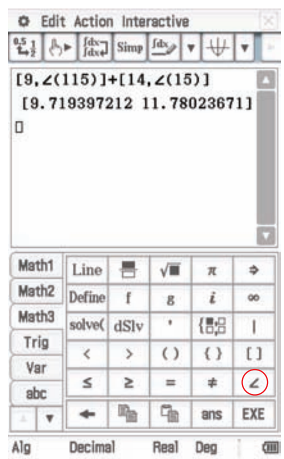
Make sure the screen is set to degrees (**Deg**) and **Decimal** at the bottom.

Press Keyboard then tap Math3 to see a button for [ ]. Vectors are entered using these square brackets with the magnitude first and then the direction, separated by ,.

The angle symbol is necessary. This is visible in the lower right part of the screen.

After tapping the symbol, enter the angle and close the brackets with).

Press EXE to obtain the result.



Round and state the result.

r ≈ (15.3, 50.5°)

EXERCISE 1.02 Addition of vectors

Concepts and techniques

1 A boat travels due east and then a different distance due north. Which of the following best describes the direction of its resultant displacement?

A north B east C north-east D north of east E south of west

2 A mass is suspended vertically by a string. The force due to gravity acting on the mass is 170 N. The force exerted by the string is 170 N vertically upwards. The resultant force exerted on the mass is:

A 170 N vertically up B 0 C 340 N vertically down
D 170 N vertically down E 340 N vertically up

3 Calculate the resultant vector in each of the following situations.

a 

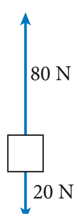
b 

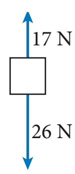
c 

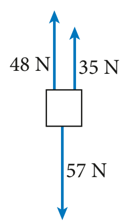
d 

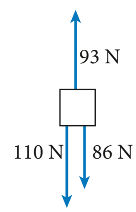
e 

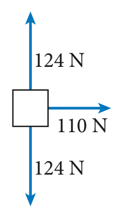
4 In each of the following situations a mass is shown with a number of forces acting on it. Calculate the resultant force in each case.

a 

b 

c 

d 

e 

5 Example 6 Given $\mathbf{w} = (6, 220°)$, $\mathbf{x} = (10, 60°)$, $\mathbf{y} = (8, 100°)$ and $\mathbf{z} = (9, -50°)$, work out the following resultant vectors, correct to one decimal place.

a $\mathbf{w} + \mathbf{y}$ b $\mathbf{y} + \mathbf{z}$ c $\mathbf{x} + \mathbf{z}$ d $\mathbf{w} + \mathbf{z}$ e $\mathbf{y} + \mathbf{x}$

6 Given $\mathbf{a} = (7, 160°)$, $\mathbf{b} = (15, 40°)$, $\mathbf{c} = (18, -120°)$ and $\mathbf{d} = (11, -50°)$, work out the following resultant vectors, correct to one decimal place.

a $\mathbf{a} + \mathbf{b}$ b $\mathbf{c} + \mathbf{b}$ c $\mathbf{d} + \mathbf{a}$ d $\mathbf{a} + \mathbf{c}$ e $\mathbf{c} + \mathbf{d}$

Reasoning and communication

7 Example 4 A ship leaves harbour and sails 12 km north to port A. From here, the ship travels 24 km due east to port B before sailing 11 km south-west to port C. Calculate the ship's resultant displacement.

8 A large rock partly buried in the ground is lifted using crowbars on opposite sides. One crowbar is exerting a vertical force of 400 N while the other is exerting a force of 300 N at an angle of 35° to the vertical. What is the total force acting and what is its direction?

9 Example 5 Sam walks from her home along a footpath to an intersection 1 km away in a north-west direction. She then walks 800 m along a road in the direction 60° north of east. Calculate Sam's resultant displacement.

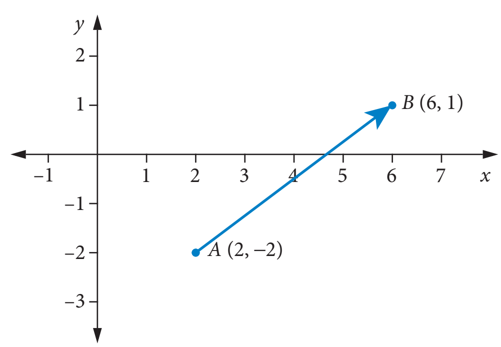
10 A boat travels 5 km NE and then 7 km at a bearing of 120°. Find its distance and direction from its starting point.

11 A bush walker walks from a base station to a viewing platform 24 km away in the direction north-east. Then the bush walker walks a further 16 km due east to a hut. Calculate the resultant displacement.

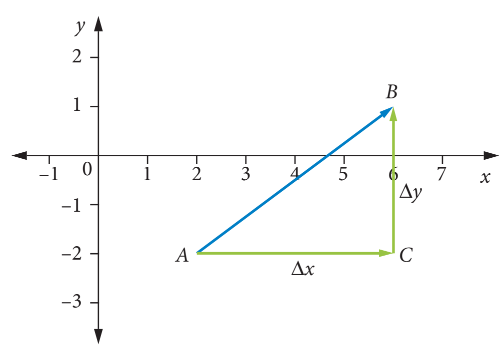
12 A bird flies from her nest looking for food. She flies at a velocity of 20 ms^{-1} in the direction of 45° south of east with respect to the air. There is a wind blowing at a constant velocity of 12 ms^{-1} in the direction 40° south of west. Calculate the resultant velocity of the bird.

1.03 COMPONENT AND POLAR FORMS OF VECTORS

Consider the displacement **AB** from $A(2, -2)$ to $B(6, 1)$ as shown on the right.



This displacement could have occurred in many different ways. It could have occurred by a movement from A to C and then from C to B, as shown in the diagram on the right. This would be an x change of $6 - 2 = 4$ and a y change of $1 - (-2) = 3$.



You can show the displacement **AB** as the x change and the y change. This gives the ordered pair (4, 3). This can be done for any vector and is called the **component form** of the vector. So, 4 is the ***x*-component** and 3 is the ***y*-component** of the vector **AB**.

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IMPORTANT

A two-dimensional geometric vector is represented in **component form** by an ordered pair, column matrix or row matrix. The first number is the x-component and the second is the y-component of the vector. They form a right-angled triangle with the vector as the hypotenuse.

Thus a two-dimensional vector $\mathbf{p}$ may be written as the row vector (x, y) or the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

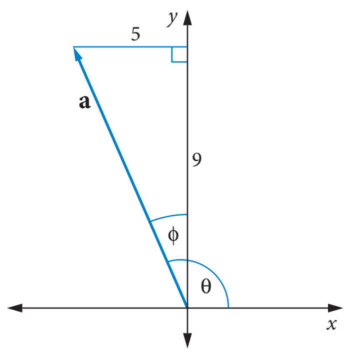
Being able to express a vector in both component and polar form is a useful skill. Trigonometry is used to convert vectors between these two representations.

Example 7

Change the vector $\mathbf{a} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

Solution

Draw $\mathbf{a}$ on the Cartesian plane using the x and y components. Draw a right-angled triangle to show the relationship between the two forms of the vector. To obtain the correct directions, care must be taken to allow for the signs of the components.



We need to calculate the magnitude a and direction θ of $\mathbf{a}$.

Use Pythagoras' theorem to find a.

$$\begin{aligned} a^2 &= 5^2 + 9^2 \\ &= 106 \\ a &\approx 10.30 \end{aligned}$$

Use the diagram to find the tan ratio.

$$\begin{aligned} \tan(\phi) &= \frac{5}{9} \\ &= 0.555\ldots \end{aligned}$$

Use $\tan^{-1}$ to find ϕ.

$$\begin{aligned} \phi &= \tan^{-1}(0.555\ldots) \\ &\approx 29.1° \end{aligned}$$

Use the diagram to find θ.

$$\begin{aligned} \theta &\approx 90° + 29.1° \\ &= 119.1° \end{aligned}$$

Write $\mathbf{a}$ in polar form.

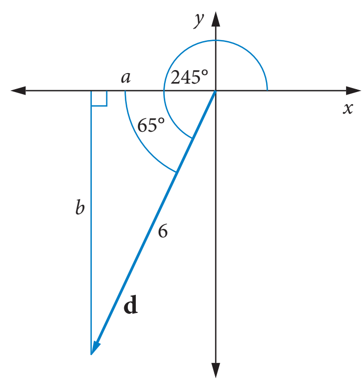
$$\mathbf{a} \approx (10.30, 119.1°)$$

Example 8

Write the vector **d**, of magnitude 6 in the direction 245°, in component form in the x and y directions.

Solution

Use the information to draw the vector. The vector is in the third quadrant.



Calculate a using the cos ratio.

$$a = 6\cos(65°)$$
$$\approx 2.54$$

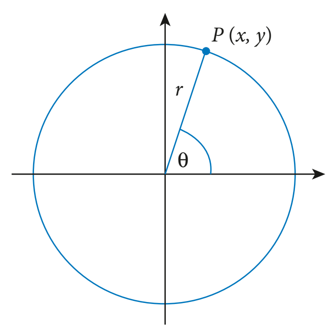
Calculate b using the sin ratio.

$$b = 6\sin(65°)$$
$$\approx 5.44$$

Write **d**, taking signs into account.

$$\mathbf{d} \approx (-2.54, -5.44)$$

For any angle θ, sin (θ), cos (θ) and tan (θ) are defined in terms of the coordinates of the point $P(x, y)$ at an angle θ from the x-axis on a circle of radius r, as shown in the diagram on the right.



$\sin(\theta) = \dfrac{y}{r}$, $\cos(\theta) = \dfrac{x}{r}$ and $\tan(\theta) = \dfrac{y}{x}$

Also $r = \sqrt{x^2 + y^2}$.

By rearranging these definitions, you get the following rules for the conversion of the polar and component forms of vectors.

IMPORTANT

For a vector **v** with polar form (r, θ) and component form (x, y):

$x = r\cos(\theta)$ $\quad$ $y = r\sin(\theta)$

$\tan(\theta) = \dfrac{y}{x}$ $\quad$ $r^2 = x^2 + y^2$

and $|\mathbf{v}| = \sqrt{x^2 + y^2}$.

When you are making vector calculations with angles, you should set the angle measure on your calculator to degrees.

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TECHNOLOGY

You can use a CAS calculator to change between the component and polar forms of a vector, using similar procedures to those shown in the previous sections.

TI-Nspire CAS

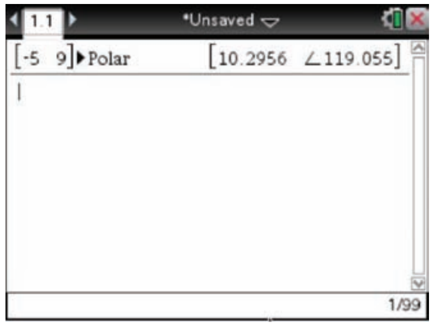
Change $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

In Document settings, check that the angle setting is in degrees and change the Calculation Mode to Approximate.

You can use menu, 7: Matrix & Vector and 1: Create to enter the vector or use the keyboard short cut

ctrl ((–) 5 , 9

Then move the cursor to the right and change to polar form using menu, 7: Matrix & Vector, C: Vector and 4: Convert to Polar.

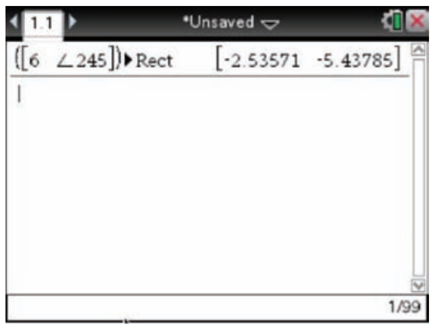


(–5, 9) ≈ (10.2956, 119.055°)

To change (6, 245°) to component form, enter the vector as before, but use the angle symbol in front of the angle. You can get this from the catalogue or by using ctrl menu and 7:Symbols.

Move the cursor to the right and use menu, 7: Matrix & Vector, C:Vector and 5:Convert to Rectangular.

You can just press enter to get rectangular coordinates because this is the default vector format.



(6, 245°) ≈ (–2.53571, –5.43785)

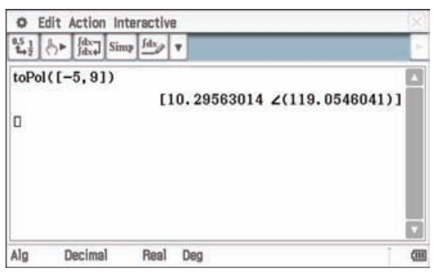
ClassPad

Change $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

Use **toPol**. First make sure that the calculator is set to **Decimal** and **Deg**.

Use the Main $\sqrt{\alpha}$ application. Tap **Action**, then **Vector**, and then **toPol**.

Enter the brackets using the [] key (located on the screen after you press Keyboard and tap Math3) then enter the vector, separating the numbers with a comma and tap **OK**.

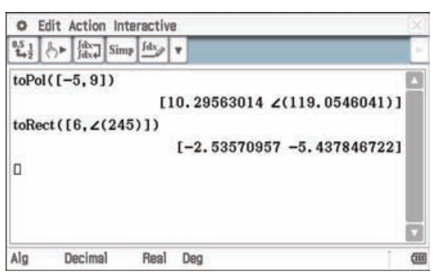


(–5, 9) ≈ (10.2956, 119.055°)

To change (6, 245°) to component form, use **toRect**. Tap **Action**, then **Vector**, and then **toRect**.

Enter the vector as before, but use the angle symbol in front of the angle and close the open (with) after entering the value. Vectors are entered using square brackets with the magnitude first and then the direction, separated by a comma ,.

Tap **OK** to perform the calculation.



(6, 245°) ≈ (–2.53571, –5.43785)

EXERCISE 1.03 Component and polar forms of vectors

WS Component and polar forms of vectors

Concepts and techniques

1 Calculate the direction of each of the following vectors, correct to 1 decimal place.

a (2, 7) b (1, 9) c (3, 6) d (5, 8)
e (−2, 9) f (6, −4) g (−8, −7) h (12, −5)

2 Calculate the magnitude of each of the following vectors, correct to 2 decimal places if necessary.

a (−3, 4) b (12, −5) c (−6, −2) d (11, 4)
e (7, −9) f (13, 8) g (−7, 12) h (13, −2)

3 Example 7 Change each of the following vectors to polar form, correct to one decimal place.

a (3, −4) b $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ c (−7, −10) d $\begin{bmatrix} 4 \\ 12 \end{bmatrix}$

e (−10, 16) f $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$ g (12, 5) h $\begin{bmatrix} -9 \\ -12 \end{bmatrix}$

i (−7, −12) i $\begin{bmatrix} 8 \\ -8 \end{bmatrix}$

4 CAS Change each of the following vectors to polar form, correct to 2 decimal places.

a (9, 12) b (11, 17) c (−9, 15) d (18, −13)
e (−20, −17) f (16.4, 8.7) g (6.37, −12.8) h (−3.91, 11.62)

5 Example 8 Change each of the following vectors to component form, correct to 2 decimal places.

a (5, 30°) b (10, 300°) c (24, 90°) d (16, 135°)
e (28, −120°) f (70, 270°) g (35, 0°) h (22, 180°)

6 CAS Change each of the following vectors to component form, correct to 2 decimal places.

a (6, 60°) b (12, 120°) c (17, 52°) d (23, 166°)
e (14, −132°) f (15, 287°) g (24, 221°) h (43, −17°)
i (4, 200°) i (8, −60°)

7 For each of the following pairs of vectors, which has the larger magnitude?

a (1, 5) or (2, 2) b (4, 7) or (3, 9) c (−2, 6) or (3, 8)
d (7, 7) or (4, −8) e (−3, −10) or (−6, 8) f (10, 4) or (6, −11)

8 If $A(5, -6)$ and $B(-2, 2)$ are the head and tail respectively of **AB**, express **AB** in polar form.

1.04 MULTIPLICATION BY SCALARS

If you wanted to make a new vector twice as big as an old one, the direction wouldn't change. You just double the magnitude. In component form, this doubles both of the components.

IMPORTANT

Multiplication by a scalar

For a vector $\mathbf{a} = (x, y)$ and a constant c, the **scalar multiple** $c\mathbf{a}$ is given by $c\mathbf{a} = (cx, cy)$.

If the vector is in polar form, i.e. $\mathbf{a} = (r, \theta)$, then for $c \geq 0$, $c\mathbf{a} = (cr, \theta)$.

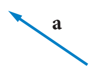
For $c < 0$, the vector is in the opposite direction, so $c\mathbf{a} = (-cr, 180° + \theta)$.

The negative sign in the last part just makes the magnitude positive.

The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.

Example 9

Vector **a** is shown on the right.



Draw vectors representing:

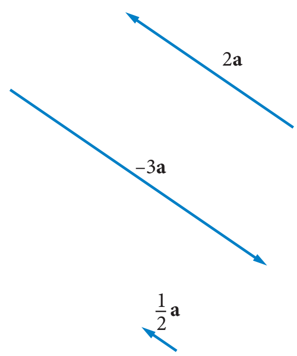
a $2\mathbf{a}$ b $-3\mathbf{a}$ c $\frac{1}{2}\mathbf{a}$

Solution

a $2\mathbf{a}$ will be in the same direction but will be twice as long as **a**.

b $-3\mathbf{a}$ will be in the opposite direction and will be three times as long as **a**.

c $\frac{1}{2}\mathbf{a}$ will be in the same direction but will be half as long as **a**.



Example 10

a Given $\mathbf{f} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, find $-3\mathbf{f}$.

b Given $\mathbf{g} = (5, 223°)$, find $-4\mathbf{g}$.

Solution

a Multiply each component of **f** by −3.

$$-3\mathbf{f} = \begin{bmatrix} -3 \times (-6) \\ -3 \times 3 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \end{bmatrix}$$

b Multiply the norm by 4 and add 180° to the angle.

$$-4\mathbf{g} = (4 \times 5, 223° + 180°)$$

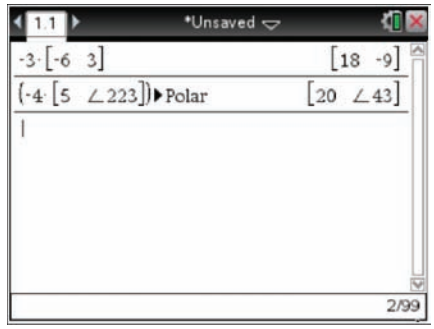
Convert the angle to a value in the domain $0 \leq \theta < 360°$.

$$= (20, 403°)$$
$$= (20, 43°)$$

You can do scalar multiplication with CAS calculators.

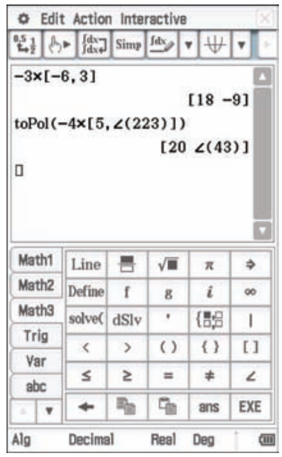
TI-Nspire CAS

Make sure your Document Settings is in degrees. Enter the vector as you did before, but put the multiplication in front and make sure the answer is given in polar form for part **b**.



ClassPad

When in polar form, you must use **toPol** so the answer remains in polar form. The calculator must be set to degrees.
In Polar form, the magnitude is given as a positive number and the angle is usually between 0° and 360°.



EXERCISE 1.04 Multiplication by scalars

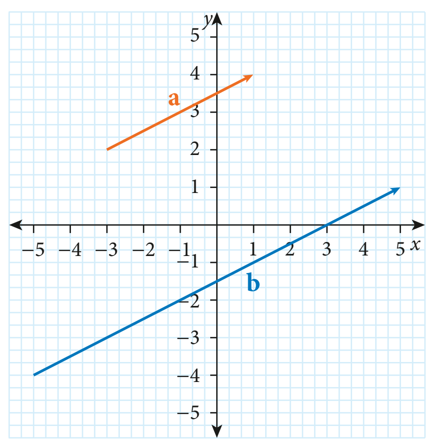
Concepts and techniques

WS Scalar multiplication

1 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = 2.5\mathbf{b}$ B $\mathbf{a} = \frac{2}{5}\mathbf{b}$ C $\mathbf{a} = \frac{1}{2}\mathbf{b}$

D $\mathbf{b} = -2\mathbf{a}$ E $\mathbf{b} = 2\mathbf{a}$



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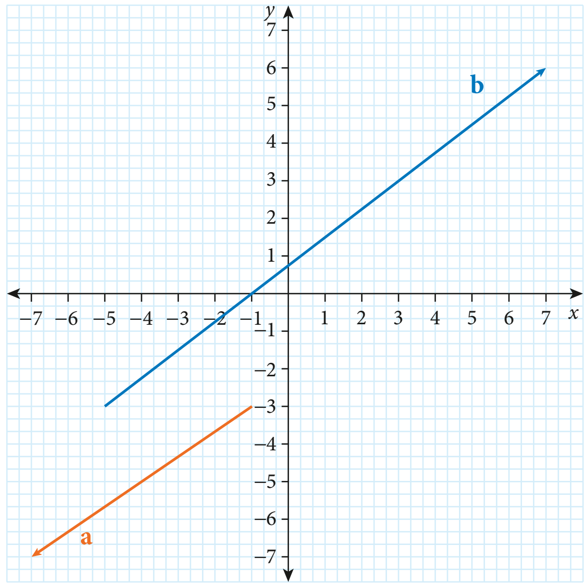
2 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = 2\mathbf{b}$ B $\mathbf{b} = -2\mathbf{a}$
C $\mathbf{a} = 3\mathbf{b}$ D $\mathbf{b} = 3\mathbf{a}$
E $\mathbf{a} = -\frac{1}{3}\mathbf{b}$



3 Example 9 If $\mathbf{b} = (6, 2)$, draw vectors representing:

a $2\mathbf{b}$ b $-\mathbf{b}$ c $3\mathbf{b}$ d $\frac{1}{2}\mathbf{b}$ e $-2\frac{1}{2}\mathbf{b}$

4 Example 10 If $\mathbf{d} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$, find:

a $3\mathbf{d}$ b $-2\mathbf{d}$ c $\frac{1}{3}\mathbf{d}$ d $6.4\mathbf{d}$
e $\frac{1}{2}\mathbf{d}$ f $-2.5\mathbf{d}$ g $\frac{2}{3}\mathbf{d}$ h $-4\mathbf{d}$

5 If $\mathbf{v} = (6, 47°)$, find

a $3\mathbf{v}$ b $-4\mathbf{v}$ c $2.5\mathbf{v}$ d $-\mathbf{v}$
e $-5\mathbf{v}$ f $10\mathbf{v}$ g $-7\mathbf{v}$ h $\frac{1}{2}\mathbf{v}$

6 If $\mathbf{a} = (2, 7)$, $\mathbf{b} = (-4, 9)$ and $\mathbf{c} = (3, -12)$, calculate:

a $4\mathbf{a}$ b $-3\mathbf{c}$ c $3.5\mathbf{b}$ d $9.7\mathbf{a}$
e $-\frac{1}{3}\mathbf{b}$ f $\frac{1}{2}\mathbf{c}$ g $\frac{3}{4}\mathbf{b}$ h $-2\mathbf{a}$

7 If $\mathbf{a} = (4, 195°)$, $\mathbf{b} = (5, 69°)$ and $\mathbf{c} = (2, 304°)$, calculate:

a $3\mathbf{a}$ b $-2\mathbf{c}$ c $1.5\mathbf{b}$ d $-4\mathbf{a}$
e $-\frac{1}{2}\mathbf{b}$ f $\frac{1}{3}\mathbf{c}$ g $-4\mathbf{b}$ h $-5\mathbf{a}$

1.05 UNIT VECTORS

You have used two different forms of geometric vectors in two dimensions—the polar form and the component form. The component form is conveniently expressed as a row or column vector. It is also useful to express it in terms of **unit vectors**.

IMPORTANT

A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is symbolised by placing a circumflex (ˆ) over the vector symbol, written as $\hat{\mathbf{p}}$.

A unit vector may be derived from any vector $\mathbf{p}$ by multiplying by $\frac{1}{|\mathbf{p}|}$.

Thus $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$.

The unit vectors in the x and y directions are given the special symbols $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. In polar form i = (1, 0°) and j = (1, 90°).

Example 11

If $\mathbf{p} = (3, -4)$, find $\hat{\mathbf{p}}$.

Solution

Find the magnitude of $\mathbf{p}$.	$\lvert\mathbf{p}\rvert = \sqrt{3^2 + (-4)^2}$
Evaluate.	$= 5$
Write the rule for the unit vector.	$\hat{\mathbf{p}} = \frac{1}{\lvert\mathbf{p}\rvert}\mathbf{p}$
Substitute in the values.	$= \frac{1}{5}(3, -4)$
Evaluate.	$\hat{\mathbf{p}} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

WS Unit vectors

A **linear combination** of vectors is a sum of scalar multiples of the vectors. Thus a linear combination of vectors $\mathbf{p}$ and $\mathbf{q}$ is of the form $a\mathbf{p} + b\mathbf{q}$, where a and b are real numbers. Any vector in two dimensions may be written as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Example 12

Write the following vectors in terms of $\mathbf{i}$ and $\mathbf{j}$.

a $\mathbf{f} = (5, -3)$ b $\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ c $\mathbf{h} = (5, 300°)$

Solution

a Write $\mathbf{f}$.	$\mathbf{f} = (5, -3)$
Express as a sum of vectors.	$= (5, 0) + (0, -3)$
Write as multiples of unit vectors.	$= 5(1, 0) + [-3(0, 1)]$
State the result.	$\mathbf{f} = 5\mathbf{i} + (-3\mathbf{j})$ $= 5\mathbf{i} - 3\mathbf{j}$

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b Write **g**.

$$\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

Express as a sum of vectors.

$$= \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Write as multiples of unit vectors.

$$= -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

State the result.

$$\mathbf{g} = -6\mathbf{i} + 0\mathbf{j}$$

c Write **h**.

$$\mathbf{h} = (5, 300°)$$

Express in component form.

$$= (5\cos(300°), 5\sin(300°))$$

Evaluate and round off.

$$\approx \left(2.5, -\frac{5\sqrt{3}}{2}\right)$$

Express as a sum of vectors.

$$= (2.5, 0) + \left(0, -\frac{5\sqrt{3}}{2}\right)$$

Write as multiples of unit vectors.

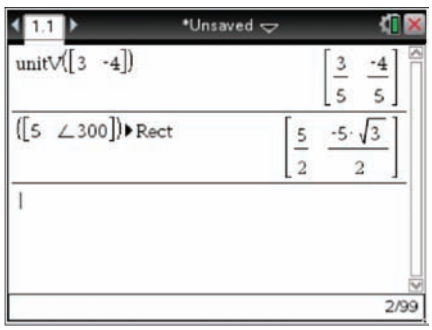
$$= 2.5(1, 0) + \left[-\frac{5\sqrt{3}}{2}(0, 1)\right]$$

State the result.

$$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$$

TI-Nspire CAS

You can calculate a unit vector by typing unitV() or using menu, 7: Matrix & Vector, C: Vector and 1: Unit Vector.
To change a vector to component form, type the vector and press enter or use menu, 7: Matrix & Vector, C: Vector and 5: Convert to rectangular.



$$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$$

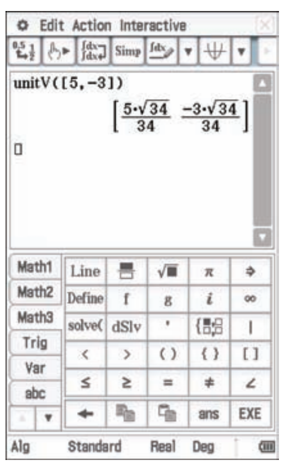
ClassPad

A unit vector can be calculated using **unitV**.

In the Main $\sqrt{\alpha}$ application, tap **Action**, then **Vector** and then **unitV**.

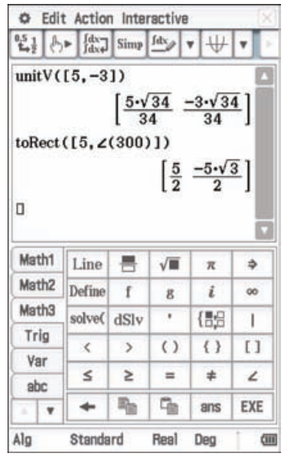
For exact answers (surds, fractions), set the calculator to **Standard**, otherwise **Decimal**.

After you have entered the vector, tap **OK**.



To write a polar vector in terms of **i** and **j**, calculate it as a column vector.

Tap **Action**, then **Vector** and then **toRect** and proceed as before.



$$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$$

EXERCISE 1.05 Unit vectors

Concepts and techniques

1 Which of the following is a unit vector?

A $(2, 90°)$ B $(1, 131°)$ C $\left(\frac{1}{2}, 180°\right)$ D $\left(1.5, \frac{\pi}{2}\right)$ E $(2, 0°)$

2 Which of the following is not a unit vector?

A $(0, 1)$ B $(1, 1)$ C $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ D $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ E $(1, 0)$

3 Example 11 Calculate the unit vector for each of the following vectors.

a $(2, 3)$ b $(4, -5)$ c $(-2, -6)$ d $(3.7, -8.2)$

e $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ f $\begin{bmatrix} -5 \\ -12 \end{bmatrix}$ g $\begin{bmatrix} 24 \\ -7 \end{bmatrix}$ h $\begin{bmatrix} 8.22 \\ 4.69 \end{bmatrix}$

i $(10, 86°)$ i $(12, -165°)$ k $\left(5, \frac{2\pi}{3}\right)$ l $\left(11, \frac{7\pi}{5}\right)$

4 Example 12 Write the following vectors as linear combinations of **i** and **j**.

a $(5, -3)$ b $(-6, 4)$ c $(-4, -7)$ d $(6, 5)$ e $(-3.2, -9.4)$

f $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ g $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$ h $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ i $\begin{bmatrix} 7.59 \\ 3.68 \end{bmatrix}$ i $\begin{bmatrix} 0.07 \\ 0.19 \end{bmatrix}$

k $(7, -74°)$ l $(9, 125°)$ m $\left(11, \frac{11\pi}{6}\right)$ n $\left(16, \frac{4\pi}{3}\right)$ o $(4.8, -119°)$

Reasoning and communication

5 Find a unit vector (in component form), that is in the opposite direction to $\mathbf{m} = 3\mathbf{i} - 4\mathbf{j}$.

6 Find a vector (in component form) of magnitude 6 that has the same direction as $\mathbf{g} = 4\mathbf{i} - 7\mathbf{j}$.

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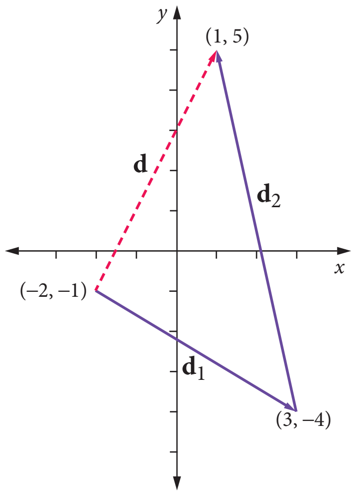
1.06 USING COMPONENTS



Consider displacements from (−2, −1) to (3, −4) and then from (3, −4) to (1, 5), as shown in the diagram on the right. The overall displacement is from (−2, −1) to (1, 5). As you have previously seen, the resultant displacement is formed as the third side of the triangle made by the first two displacements, with the vectors placed head-to-tail.

You can find the resultant displacement without using a diagram and the triangle of vectors.

In component form, the displacements are $\mathbf{d}_1 = (5, -3)$ and $\mathbf{d}_2 = (-2, 9)$ and the resultant displacement is $\mathbf{d} = (3, 6)$.

You can simply add the components of $\mathbf{d}_1$ and $\mathbf{d}_2$ as shown below.

$$\begin{aligned}\mathbf{d}_1 + \mathbf{d}_2 &= (5, -3) + (-2, 9)\\ &= (5 + (-2), -3 + 9)\\ &= (3, 6)\\ &= \mathbf{d}\end{aligned}$$

When displacements are given in polar form, we must either draw a diagram and use the triangle of vectors or convert vectors to component form before adding.

IMPORTANT

The **sum** (or **resultant**) of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is given by

$$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2)$$

In terms of the unit vectors **i** and **j**, if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= x_1\mathbf{i} + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{j}\\ &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}\end{aligned}$$

Example 13

Find the sum of the vectors:

a (3, −7) and (−11, 8) **b** $\begin{bmatrix}7\\3\end{bmatrix}$ and $\begin{bmatrix}5\\-4\end{bmatrix}$ **c** $5\mathbf{i} + 7\mathbf{j}$ and $-9\mathbf{i} - 4\mathbf{j}$

Solution

a The vectors are in component form, so add the components.

$$\begin{aligned}(3, -7) + (-11, 8) &= (3 + (-11), -7 + 8)\\ &= (-8, 1)\end{aligned}$$

b The vectors are in component form, so add the components.

$$\begin{aligned}\begin{bmatrix}7\\3\end{bmatrix} + \begin{bmatrix}5\\-4\end{bmatrix} &= \begin{bmatrix}7+5\\3+(-4)\end{bmatrix}\\ &= \begin{bmatrix}12\\-1\end{bmatrix}\end{aligned}$$

c The vectors are expressed in terms of **i** and **j**, so add the **i** and **j** parts separately.

$$\begin{aligned}5\mathbf{i} + 7\mathbf{j} + (-9\mathbf{i} - 4\mathbf{j}) &= (5 + -9)\mathbf{i} + [7 + (-4)]\mathbf{j}\\ &= -4\mathbf{i} + 3\mathbf{j}\end{aligned}$$

You looked at scalar multiples of vectors in the previous section.

IMPORTANT

The difference of vectors

The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.

The difference of vectors is found by addition of the negative.

Thus $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Example 14

Given that $\mathbf{a} = (6, -4)$ and $\mathbf{b} = (-5, 2)$, find:

a $\mathbf{a} - \mathbf{b}$ b $2\mathbf{a} - 3\mathbf{b}$

Solution

a Use the definition of subtraction.

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

Substitute in the known values.

$= (6, -4) + [-(-5, 2)]$

Evaluate.

$= (6 + 5, -4 - 2)$
$= (11, -6)$

b Use the definition of scalar multiplication.

$2\mathbf{a} - 3\mathbf{b} = 2(6, -4) - 3(-5, 2)$

Evaluate.

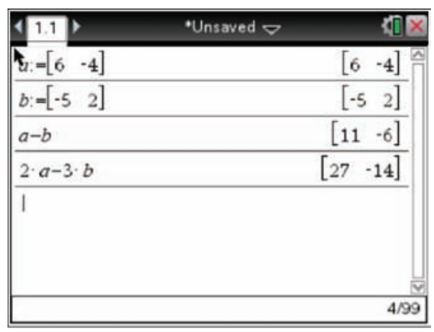
$= (12, -8) - (-15, 6)$

Use the definition of subtraction and evaluate.

$= (12, -8) + [-(-15, 6)]$
$= (27, -14)$

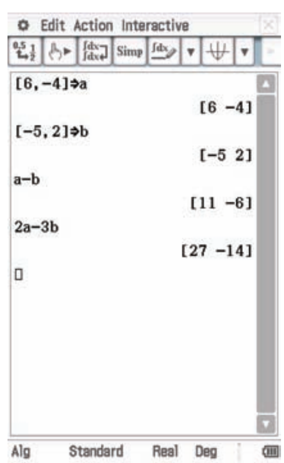
TI-Nspire CAS

You can name vectors and add, subtract, and multiply them by scalars just as you would do for pronumerals.
To define a vector, use the := symbol (ctrl ⁝=) or use Define.
The screen on the right first defines vectors **a** and **b**, and then calculates **a** – **b** and 2**a** – 3**b**.



ClassPad

You can name vectors and add, subtract, and multiply them by scalars just as you would do for pronumerals.
To define a vector as **a**, use ⇒ a.
Press Keyboard then tap Math3 for the keyboard with the square brackets and ⇒, and tap abc for the letters keyboard.
The screen on the right first defines vectors **a** and **b**, and then calculates **a** – **b** and 2**a** – 3**b**.



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EXERCISE 1.06 Using components

WS
Addition of vectors using components

Concepts and techniques

1 Example 13 Find the sum of the following pairs of vectors.

a $(-4, 3)$ and $(-9, 10)$
b $(7, -9)$ and $(11, -5)$
c $(-13, 6)$ and $(8, -7)$
d $(3.8, -4.5)$ and $(6.2, -7.3)$
e $(-1.07, 0.35)$ and $(5.24, -4.57)$
f $\left(3\frac{1}{2}, -2\frac{4}{5}\right)$ and $\left(-1\frac{3}{4}, 5\frac{2}{3}\right)$

2 Find the sum of the following pairs of vectors.

a $\begin{bmatrix} 14 \\ -11 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$
b $\begin{bmatrix} -5 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ 7 \end{bmatrix}$
c $\begin{bmatrix} 13 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -8 \end{bmatrix}$
d $\begin{bmatrix} -3.6 \\ -8.7 \end{bmatrix}$ and $\begin{bmatrix} 6.9 \\ -4.6 \end{bmatrix}$
e $\begin{bmatrix} 2.07 \\ -4.66 \end{bmatrix}$ and $\begin{bmatrix} -0.97 \\ -1.78 \end{bmatrix}$
f $\begin{bmatrix} 2\frac{1}{2} \\ -3\frac{4}{9} \end{bmatrix}$ and $\begin{bmatrix} 3\frac{3}{4} \\ -2\frac{1}{3} \end{bmatrix}$

3 Find the sum of the vectors below.

a $5\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{i} - 7\mathbf{j}$
b $-8\mathbf{i} + 7\mathbf{j}$ and $-9\mathbf{i} + 3\mathbf{j}$
c $-11\mathbf{i} - 4\mathbf{j}$ and $6\mathbf{i} + 9\mathbf{j}$
d $5.8\mathbf{i} - 6.7\mathbf{j}$ and $-9.2\mathbf{i} - 5.3\mathbf{j}$
e $-2.14\mathbf{i} + 1.79\mathbf{j}$ and $-6.09\mathbf{i} + 3.36\mathbf{j}$
f $-4\frac{2}{5}\mathbf{i} + 3\frac{3}{8}\mathbf{j}$ and $-5\frac{7}{10}\mathbf{i} + 6\frac{1}{4}\mathbf{j}$

4 Example 14 Given that $\mathbf{a} = (1, 4)$, $\mathbf{b} = (-7, 8)$, $\mathbf{c} = (2, 4)$, $\mathbf{d} = (5, -2)$ and $\mathbf{e} = (-6, 1)$, find:

a $\mathbf{c} + \mathbf{e}$
b $\mathbf{b} + \mathbf{c} + \mathbf{d}$
c $5\mathbf{a}$
d $-2\mathbf{e}$
e $3\mathbf{a} + 2\mathbf{e}$
f $4\mathbf{b} - 2\mathbf{a}$
g $\mathbf{a} - \mathbf{c} - \mathbf{e}$
h $7\mathbf{c} - 7\mathbf{a} + \mathbf{d} + 2\mathbf{e}$
i $7\mathbf{a} + 5\mathbf{b} + \mathbf{e}$
i $2\mathbf{d} - 7\mathbf{c}$

5 Change the following to component form, find the result, and then change back to polar form.

a $(6, 20°) + (9, 55°)$
b $(25, 120°) + (16, 80°)$
c $(7, 30°) - (9, 100°)$
d $(105, 300°) - (95, 60°)$
e $(6, 70°) - (6, 10°)$

Reasoning and communication

6 Find the displacement vector for each of the following movements.

a $A(3, 4)$ to $B(5, 11)$
b $R(-2, 1)$ to $B(-7, -2)$
c $M(-3, -8)$ to $B(-9, 1)$
d $Q(1, 8)$ to $P(1, -10)$
e $B(-2, -5)$ to $X(6, 9)$
f $F(-1, 4)$ to $G(7, -7)$

7 If $\mathbf{a} = (x_1, y_1)$ and $c < 0$, prove that $|c\mathbf{a}| = -c|\mathbf{a}|$.

8 If $\mathbf{a} = (x_1, y_1)$, $\mathbf{b} = (x_2, y_2)$ and c is a scalar, prove that $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$.

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1.07 VECTOR PROPERTIES

In the previous sections of this chapter you discovered a number of vector properties. In this section, some of the more important properties will be formalised.

IMPORTANT

Vector properties for addition and multiplication by a scalar

The following properties are defined for vectors **a**, **b** and **c** and scalars r and s.

1 Vector addition is **associative**.

$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

2 Vector addition is **commutative**.

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

3 There is an **additive identity**.

The additive identity has all elements equal to zero and is called the **zero vector**, written as **0**.

$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

4 There is an additive **inverse**.

The additive inverse of $\mathbf{a} = (x_1, y_1)$ is $-\mathbf{a} = (-x_1, -y_1)$.

In polar form, $\mathbf{a} = (r, \theta)$ has the additive inverse $-\mathbf{a} = (r, \theta + 180°)$.

$\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = \mathbf{0}$

5 **Cancellation laws** hold for multiplication of a vector by a scalar.

$r\mathbf{a} = r\mathbf{b} \Leftrightarrow \mathbf{a} = \mathbf{b}\ (r \neq 0)$

$r\mathbf{a} = s\mathbf{a} \Leftrightarrow r = s\ (\mathbf{a} \neq \mathbf{0})$

6 **Distributive laws** hold for multiplication over addition.

$(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

$r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

7 An **associative law** holds for multiplication and multiplication by a scalar.

$(rs)\mathbf{a} = r(s\mathbf{a})$

The properties outlined above can be demonstrated through the use of particular vectors and scalars.

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Example 15

Demonstrate that vector addition is commutative for vectors $\mathbf{u} = (5, -1)$ and $\mathbf{v} = (-6, 8)$.

Solution

Write down what needs to be demonstrated.

Need to show: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Calculate $\mathbf{u} + \mathbf{v}$.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (5, -1) + (-6, 8)\\ &= (5 - 6, -1 + 8)\\ &= (-1, 7)\end{aligned}$$

Calculate $\mathbf{u} + \mathbf{v}$.

$$\begin{aligned}\mathbf{v} + \mathbf{u} &= (-6, 8) + (5, -1)\\ &= (-6 + 5, 8 - 1)\\ &= (-1, 7)\end{aligned}$$

You have shown that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Vector addition is commutative for vectors $\mathbf{u}$ and $\mathbf{v}$.

The proofs of the previously described vector properties follow the same approach as outlined in Example **15**, but use general rather than particular values.

WS

Vector properties

Example 16

Prove the distributive law for multiplication of a scalar over vector addition.

Solution

Write what is required to be proved (RTP).

RTP: $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

Identify the vectors and the scalar quantity.

Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ and let r be a scalar quantity.

Substitute the values in the LHS of the RTP statement.

$r(\mathbf{a} + \mathbf{b}) = r(x_1 + x_2, y_1 + y_2)$

Expand the brackets

$= (rx_1 + rx_2, ry_1 + ry_2)$

Regroup like terms.

$= (rx_1, ry_1) + (rx_2, ry_2)$

Factorise.

$= r(x_1, y_1) + r(x_2, y_2)$

Substitute in the defined terms.

$= r\mathbf{a} + r\mathbf{b}$ **QED**

EXERCISE 1.07 Vector properties

Concepts and techniques

1 If **c** and **d** are vectors, which of the following statements is not necessarily true?

A $\mathbf{c} + \mathbf{d} = \mathbf{d} + \mathbf{c}$ B $\mathbf{d} + \mathbf{0} = \mathbf{d}$ C $\mathbf{c} + (-\mathbf{c}) = \mathbf{0}$

D $\mathbf{c} - \mathbf{d} = \mathbf{d} - \mathbf{c}$ E $\mathbf{0} + \mathbf{c} = \mathbf{c}$

2 If **p**, **q** and **r** are vectors, and m and n are scalar quantities, which of the following statements is not necessarily true?

A $(n + m)\mathbf{r} = n\mathbf{r} + m\mathbf{r}$ B $\mathbf{rp} + \mathbf{rq} = \mathbf{r}(\mathbf{p} + \mathbf{q})$ C $m\mathbf{q} + n\mathbf{q} = (m + n)\mathbf{q}$

D $m(\mathbf{p} + \mathbf{q}) = m\mathbf{p} + m\mathbf{q}$ E $(mn)\mathbf{r} = m(n\mathbf{r})$

3 Find the additive inverse of each of the following vectors.

a $\mathbf{m} = (-5, 8)$ b $\mathbf{d} = (-11, -4)$ c $\mathbf{r} = (0, -6)$

d $\mathbf{q} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ e $\mathbf{g} = \begin{bmatrix} 14 \\ 3 \end{bmatrix}$ f $\mathbf{h} = \begin{bmatrix} -12 \\ 11 \end{bmatrix}$

g $\mathbf{a} = 4\mathbf{i} - 9\mathbf{j}$ h $\mathbf{p} = \frac{2}{3}\mathbf{i} + \frac{4}{5}\mathbf{j}$ i $\mathbf{n} = -0.24\mathbf{i} - 1.06\mathbf{j}$

4 Example 15 Demonstrate that vector addition is commutative for each pair of vectors below.

a $\mathbf{m} = (-3, -5)$ and $\mathbf{p} = (2, 9)$ b $\mathbf{q} = (10, -7)$ and $\mathbf{d} = (-8, 4)$

c $\mathbf{f} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ d $\mathbf{u} = \begin{bmatrix} 15 \\ -6 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$

e $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = -5\mathbf{i} + 7\mathbf{j}$ f $\mathbf{e} = -11\mathbf{i} - 6\mathbf{j}$ and $\mathbf{n} = 13\mathbf{i} + 8\mathbf{j}$

5 Demonstrate that vector addition is associative for each set of vectors below.

a $\mathbf{a} = (-2, 6)$, $\mathbf{b} = (1, -7)$ and $\mathbf{c} = (-8, 0)$

b $\mathbf{p} = (-1, 9)$, $\mathbf{q} = (-4, -6)$ and $\mathbf{r} = (5, -3)$

c $\mathbf{t} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -8 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

d $\mathbf{k} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

e $\mathbf{w} = -\mathbf{i} - 3\mathbf{j}$, $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{s} = 9\mathbf{i} - 6\mathbf{j}$

f $\mathbf{e} = -4\mathbf{i} + 11\mathbf{j}$, $\mathbf{d} = -\mathbf{i} - 3\mathbf{j}$ and $\mathbf{f} = 8\mathbf{i} - 4\mathbf{j}$

6 Demonstrate that the distributive law for multiplication of a vector over scalar addition holds true for each grouping of vector and scalars shown below.

a $\mathbf{h} = (1, -2)$, $w = 3$ and $g = 2$ b $\mathbf{d} = (-12, 18)$, $a = \frac{1}{2}$ and $b = \frac{2}{3}$

c $\mathbf{a} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $m = 4$ and $n = 5$ d $\mathbf{b} = \begin{bmatrix} -16 \\ -8 \end{bmatrix}$, $r = 0.5$ and $s = 0.25$

e $\mathbf{q} = 5\mathbf{i} - \mathbf{j}$, $c = 2$ and $d = 6$ f $\mathbf{p} = -12\mathbf{i} - 36\mathbf{j}$, $k = \frac{1}{3}$ and $v = \frac{3}{4}$

Reasoning and communication

7 Example 16 Prove that vector addition is associative.

8 Prove that vector addition is commutative.

9 Given that **p** and **q** are vectors and k is a scalar quantity, prove that $k(\mathbf{p} + \mathbf{q}) = k\mathbf{p} + k\mathbf{q}$.

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1.08 APPLICATIONS OF VECTORS

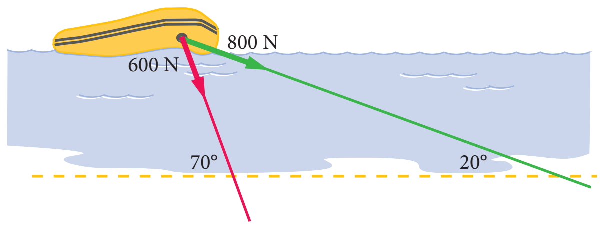
Many situations, such as displacement, velocity, force and acceleration, involve vector quantities. The techniques dealt with in this chapter can be used to solve real world problems involving vectors.

Example 17

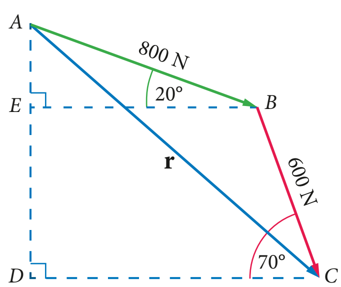
Two ropes are tied to the prow of a rubber boat and are pulling the boat towards the beach. The first rope is being pulled by two people at an angle of 20° to the shoreline, with a force of 800 N (newtons). The second rope is being pulled in the same general direction by a lifeguard, with a force of 600 N at an angle of 70° to the shoreline. What is the total force acting on the boat and in what direction is it acting?

Solution

Draw a diagram to show the forces.



Draw the forces head-to-tail to view the triangle of forces ABC at a larger scale, and complete the triangle with the resultant.
Since $EBCD$ is a quadrilateral, $\angle EBC = 110°$, so $\angle ABC = 130°$.



Use the cosine rule to find r, stating the rule first.

$$\begin{aligned} r^2 &= AB^2 + BC^2 - 2 \times AB \times BC \cos(\angle ABC) \\ &= 800^2 + 600^2 - 2 \times 800 \times 600 \cos(130°) \\ &= 1\,617\,076.105... \end{aligned}$$

Keep the exact answer in your calculator.

$$r = 1271.643...\text{ N}$$

Use the sine rule to find $\angle BCA$.

$$\frac{\sin(\angle BCA)}{AB} = \frac{\sin(\angle ABC)}{AC}$$

Rearrange.

$$\sin(\angle BCA) = \frac{AB \times \sin(\angle ABC)}{AC}$$

Use the exact answer from your calculator.

$$\begin{aligned} &= \frac{800 \times \sin(130°\ )}{1271.643...} \\ &= 0.481\,9924... \end{aligned}$$

Use $\sin^{-1}$ to find $\angle BCA$.	$\angle BCA = 28.811...°$
Calculate the angle of the resultant with the shoreline.	Resultant angle $= 70° - 28.811...°$ $= 41.188...°$
State the answer in an appropriate form.	The total force acting on the boat is about 1272 N, at an angle of 41.2° to the shoreline.

The change in a quantity from x_1 to x_2, can be shown as $\Delta x = x_2 - x_1$. This also applies to vector quantities, but you need to use vector addition. So, for velocity, $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

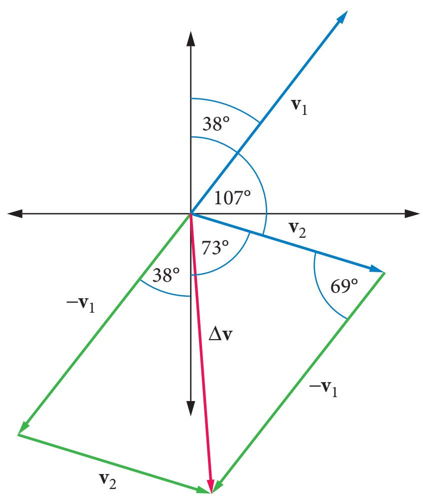
Example 18

The velocity of a boat changes from 25 knots at N 38° E to 15 knots at a bearing of 107°. What is the change in velocity?

iStockphoto/Ian Hamilton

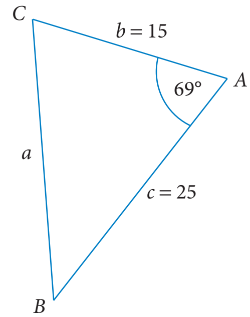
Solution

Sketch a diagram to represent the situation.
Let $\mathbf{v}_1$ be the velocity 25 knots at N 38° E.
Let $\mathbf{v}_2$ be the velocity 15 knots at 107°.
Put $-\mathbf{v}_1$ on the diagram.
Rearrange $-\mathbf{v}_1$ and $\mathbf{v}_2$ on the diagram so that the change of velocity can be found.
Find the angle between the vectors.
The obtuse angle in the parallelogram is 111° (38° + 73°), so the acute angle is 69°.



Express the change of velocity in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.	$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ $= \mathbf{v}_2 + -\mathbf{v}_1$

We could proceed using either the component or polar form of the vectors.
We will use the polar form.
Draw a diagram to show the information of interest with a, b and c for the magnitudes of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\Delta\mathbf{v}$.



Use the cosine rule to find a.
Substitute in the known values.

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$= 15^2 + 25^2 - 2 \times 15 \times 25 \times \cos(69°)$$

Evaluate and retain the unrounded result. $a = 24.108\ldots$

Use the sine rule to find C. $\dfrac{\sin(C)}{c} = \dfrac{\sin(A)}{a}$

Substitute in the known values. $\dfrac{\sin(C)}{25} = \dfrac{\sin(69°)}{24.108\ldots}$

Rearrange and evaluate. $\sin(C) = 0.9680\ldots$

Use $\sin^{-1}$ to find C. $C \approx 75.5°$

Calculate the bearing of the resultant. $\text{Bearing} \approx 107° + 75.5°$

Evaluate. $= 182.5°$

If you refer back to the original diagram, you can see that the sketch was slightly incorrect because it shows the bearing of $\Delta\mathbf{v}$ as less than 180° (107° + 73°). So you can see that as long as the sketch is roughly correct to start with, a slight error of this nature doesn't matter.

State the result. The change in velocity is about 24.1 knots at a bearing of 182.5°.

EXERCISE 1.08 Application of vectors

Reasoning and communication

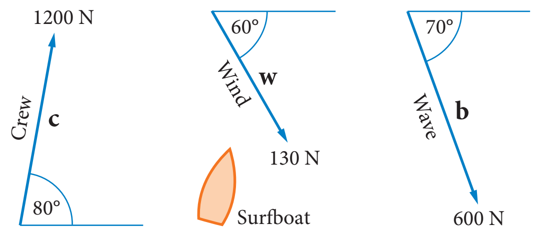
WS Vector applications

1 Example 17 Two winches are being used to pull out a bogged car. One winch is on the driver's side at an angle of 32° to the forward direction and the larger winch is on the passenger's side at an angle of 39° to the forward direction. The first is exerting a force of 6000 N and the other is exerting a force of 9000 N. What is the total force on the car, and in what direction does it act?

2 An orienteer runs 5 km at a bearing of 125° and then 3 km at a bearing of 205°. Find her distance and bearing from the starting point.

3 A boat is being pulled along a canal by two people using ropes. One person is on the left bank pulling at an angle of 30° to the bank with a force of 400 N. The other person is on the right bank pulling at an angle of 45° to the bank. The boat is moving straight along the canal. With what force is the second person pulling?

4 A surfboat at a lifesavers' regatta is putting out from the beach. The directions of the forces acting on the boat are shown below, with the crew exerting a force of 1200 N, the wind a force of 130 N and a breaking wave a force of 600 N. Find the direction and magnitude of the resultant force on the surfboat.




Newspix/Scott Fletcher

5 Example 18 A car travelling at 20 m/s south changes velocity to 18 m/s east. Find the change in velocity.

6 Police called to an emergency are travelling directly west along a main road and slow down to 60 km h^{-1} as they approach an intersection, where they turn right (by 90°). What is the change of velocity of the car?

7 An aircraft travelling at 140 knots at a bearing of 197° changes direction to a bearing of 116° at the same speed to approach the runway from the seaward side. Find the change in velocity.

8 What is the change of velocity when a cyclist in a road race changes velocity from 15 m/s NW to 12 m/s N?

9 Find the change of velocity when an object travelling at v m/s changes direction by θ° without changing speed.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CHAPTER SUMMARY
BASIC VECTORS

1

- A **scalar** quantity is a **magnitude** expressed by a single number, but a **geometric vector** has both a magnitude (**norm**) and a **direction**. A vector is usually denoted by a lower case bold letter, such as **a**, and may be shown by a **directed line segment** whose length represents the magnitude $|\mathbf{a}| = a$.
- A **displacement** vector showing a change of position from A to B may be written as **AB** **or** $\overrightarrow{AB}$. The **position vector** of a point P is the vector $\mathbf{p} = \mathbf{OP}$. O is the **initial point** and P is the **terminal point** of the vector.
- The **polar form** of a **two-dimensional vector** $\mathbf{v} = (r, \theta)$ shows the magnitude r and the direction θ of the vector. θ is the angle in the positive direction (anticlockwise) from the x-axis. The **component form** (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$ shows the vector as **components** in the x- and y-directions.

  The polar and component forms of the vector **v** are related by:

  $$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

  $$\tan(\theta) = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

  $$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

- The sum $\mathbf{a} + \mathbf{b}$ is also called the **resultant** of **a** and **b**. The resultant vector may be found geometrically using the **triangle of vectors**. This is also called the head-to-tail method. Addition of vectors can also be done geometrically using a **parallelogram** of vectors. For a vector $\mathbf{a} = (x_1, y_1)$ and a constant c, the **scalar multiple** $c\mathbf{a}$ is defined as $c\mathbf{a} = (cx_1, cy_1)$. If the vector is in polar form, i.e. $\mathbf{a} = (r, \theta)$, the scalar multiple $c\mathbf{a}$ is (cr, θ) for $c \geq 0$, and $(-cr, 180° + \theta)$ for $c < 0$. The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.
- A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector **p** is symbolised as $\hat{\mathbf{p}}$, where $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$.
- The unit vectors in the x- and y-directions are **i** and **j**, so $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

  $\mathbf{i} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

  Any vector can be shown as a linear combination of **i** and **j**, so $\mathbf{v} = r\mathbf{i} + s\mathbf{j}$ for some $r, s \in \boldsymbol{R}$.
- The **sum** (or **resultant**) of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is defined by

  $$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2).$$

- Addition of vectors is **associative** and **commutative**, there is an **identity** vector $\mathbf{0} = (0, 0)$ and any vector **a** has an additive inverse $-\mathbf{a}$.

  Cancellation, commutative, associative and **distributive laws** hold for vector addition and scalar multiplication.

  $$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

  $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

  $r\mathbf{a} = r\mathbf{b} \rightarrow \mathbf{a} = \mathbf{b}\ (r \neq 0)$ and
  $r\mathbf{a} = s\mathbf{a} \rightarrow r = s\ (\mathbf{a} \neq \mathbf{0})$

  $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + s\mathbf{a}$ and
  $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$
  $r(s\mathbf{a}) = (rs)\mathbf{a}$

  The change in a vector from $\mathbf{v}_1$ to $\mathbf{v}_2$ can be written as $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

CHAPTER REVIEW
BASIC VECTORS

Multiple choice

1 Example 4 What is the direction of the position vector of $A(-2, -2)$?

A $-45°$ B $45°$ C $135°$ D $225°$ E $315°$

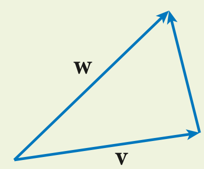
2 Examples 1, 2 An aircraft is travelling north-west at 128 km h^{-1}. Which of the following polar forms best describes the velocity of the vessel?

A (128, 135°) B (128, 45°) C (128, 315°)
D (128 km h^{-1}, 225°) E (128, −135°)

3 Example 3 A ball falls vertically downwards 11 m and rebounds to a height of 7 m. Which of the following best describes the resultant displacement of the ball?

A 18 m B 4 m down C 7 m up D 4 m E 4 m up

4 Example 4 This diagram shows vectors **w**, **v** and a third vector. Which of the following best describes the third vector?

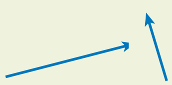


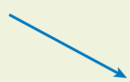
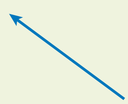
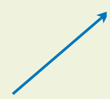
A $\mathbf{w} + \mathbf{v}$ B $\mathbf{v} + (-\mathbf{w})$ C $-\mathbf{w} + \mathbf{v}$
D $\mathbf{w} - \mathbf{v}$ E $\mathbf{v} + \mathbf{w}$

5 Example 10 If $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$, which of the following vectors is opposite in direction to **b**?

A $2\mathbf{i} + 5\mathbf{j}$ B $\begin{bmatrix} -4 \\ 10 \end{bmatrix}$ C $-\mathbf{w} + \mathbf{v}$ D $\mathbf{w} - \mathbf{v}$ E $\mathbf{v} + \mathbf{w}$

6 Example 4 This diagram shows vectors **a** and **b**. Which of the following best represents the resultant of **a** and **b**?



A B C  D  E 

7 Example 11 Which of the following is a unit vector?

A $\left(\frac{1}{2}, 90°\right)$ B (0, 0) C (1.5, 90°) D (1, 248°) E (0.5, 180°)

8 Example 11 Which of the following is not a unit vector?

A (1, 1) B $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ C (1, 0) D $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ E (−1, 0)

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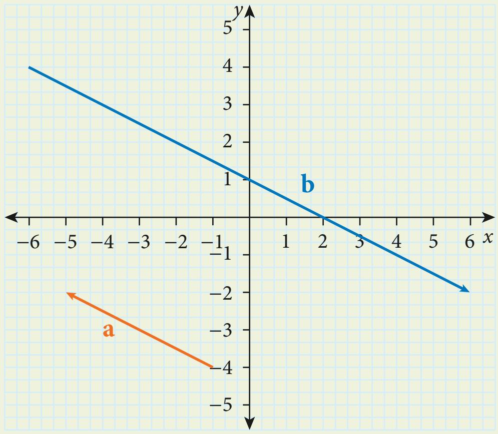
9 Example 9 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = -2\mathbf{b}$ B $\mathbf{b} = 2\mathbf{a}$

C $\mathbf{a} = -\frac{1}{3}\mathbf{b}$ D $\mathbf{b} = 3\mathbf{a}$

E $\mathbf{a} = \frac{1}{2}\mathbf{b}$



Short answer

10 Examples 1, 2 Use directed line segments to represent the following displacements on the same set of axes.

a 5 units north-east

b 6 units in the direction 150°

11 Example 3 Find the displacement vectors for each of the following movements.

a $D(3, -4)$ to $E(1, 2)$

b $T(-3, -5)$ to $S(5, 0)$

c $G(1, 1)$ to $J(-3, 5)$

d $R(2, 4)$ to $K(0, 8)$

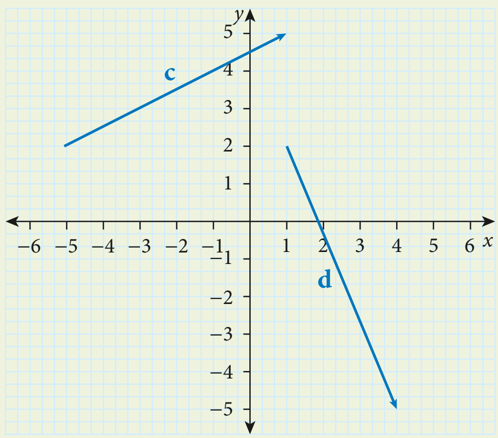
e $Y(-6, 3)$ to $Z(1, -9)$

12 Example 4 Vectors **c** and **d** are shown on the right. Calculate the resultant of each of the following.

a $\mathbf{c} + \mathbf{d}$ b $\mathbf{d} + \mathbf{c}$

c $\mathbf{d} + -\mathbf{d}$ d $\mathbf{c} + -\mathbf{d}$

e $\mathbf{c} + \mathbf{c} + \mathbf{d}$



13 Example 7 Convert the following vectors to polar form.

a $(5, 8)$ b $(-3, 4)$ c $(-6, -10)$ d $(9, -3)$ e $(-4, 0)$

14 Example 8 Convert the following vectors to component form.

a $(3, 30°)$ b $(8, 150°)$ c $(4, 300°)$ d $(7, 45°)$ e $(9, 230°)$

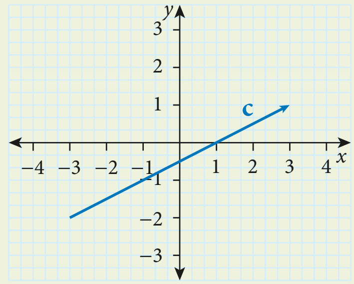
15 Example 7 Convert the following vectors to polar form.

a $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ b $\begin{bmatrix} -2 \\ -6 \end{bmatrix}$ c $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ d $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ e $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$

16 Example 7 Find the norm of each of the following vectors.

a $(24, 10)$ b $(-3, -4)$ c $\begin{bmatrix} -8 \\ 15 \end{bmatrix}$ d $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ e $(10, 12)$

17 Example 9 Vector **c** is shown on the right. Draw vectors representing $-2\mathbf{c}$ and $\frac{1}{3}\mathbf{c}$.



18 Example 10 If $\mathbf{a} = \begin{bmatrix} 12 \\ -9 \end{bmatrix}$, find $-6\mathbf{a}$.

19 Example 11 If $\mathbf{m} = (7, -24)$ find $\hat{\mathbf{m}}$.

20 Example 12 Write the following vectors in terms of **i** and **j**.

a $(1, -3)$ b $(4, 5)$ c $(-2, 5)$ d $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

e $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$ f $(6, 56°)$ g $(8, 200°)$ h $(3, 180°)$

i $(10, 142°)$ i $(5, 120°)$

21 Example 13 Find the resultant of each of the following pairs of vectors.

a $4\mathbf{i} - 7\mathbf{j}$ and $8\mathbf{i} + 2\mathbf{j}$ b $(3, -8)$ and $(-9, 6)$ c $\begin{bmatrix} -1 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -6 \end{bmatrix}$

22 Example 14 Given $\mathbf{a} = (3, 5)$, $\mathbf{b} = (-2, -3)$ and $\mathbf{c} = (-1, 4)$, find:

a $\mathbf{a} + \mathbf{c}$ b $\mathbf{b} - \mathbf{a}$ c $3\mathbf{b} + 2\mathbf{c}$ d $5\mathbf{a} - 3\mathbf{b}$ e $\mathbf{a} + \mathbf{b} + \mathbf{c}$

23 Example 14 Given $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, find:

a $\mathbf{x} - \mathbf{y}$ b $2\mathbf{x} + 3\mathbf{y}$ c $\mathbf{y} + 3\mathbf{z}$ d $\mathbf{z} - 2\mathbf{x}$ e $\mathbf{x} + \mathbf{z}$

24 Example 6 Given $\mathbf{p} = (5, 85°)$, $\mathbf{q} = (7, 146°)$ and $\mathbf{r} = (6, 212°)$, find the following without changing to component form.

a $\mathbf{p} + \mathbf{q}$ b $\mathbf{q} + \mathbf{r}$ c $\mathbf{r} + \mathbf{p}$ d $\mathbf{p} - \mathbf{r}$ e $\mathbf{r} - \mathbf{q}$

25 Example 14 Given $\mathbf{f} = (5, 150°)$, $\mathbf{g} = (8, 60°)$ and $\mathbf{h} = (9, 225°)$, find the following by changing to component form and expressing the answer in polar form.

a $\mathbf{f} - \mathbf{g}$ b $\mathbf{g} - \mathbf{h}$ c $\mathbf{g} + \mathbf{h}$ d $\mathbf{f} + \mathbf{g}$ e $\mathbf{h} + \mathbf{g} + \mathbf{f}$

26 Example 15 Demonstrate that vector addition is associative for $\mathbf{r} = (2, -5)$, $\mathbf{g} = (-2, -7)$ and $\mathbf{k} = (3, 2)$.

27 Example 16 If $\mathbf{m} = (x_1, y_1)$ and $\mathbf{n} = (x_2, y_2)$, prove that $\mathbf{m} + \mathbf{n} = \mathbf{n} + \mathbf{m}$.

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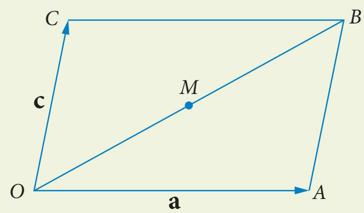
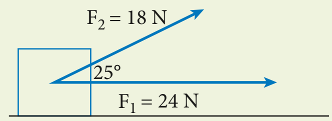
Application

28 *OABC* is a parallelogram and *M* is the midpoint of **OB**. Taking *O* as the origin, the position vectors of *A* and *C* are **a** and **c** respectively. Express each of the following in terms of **a** and **c**.

a **BC**
b **AC**
c **OB**
d **OM**
e **AM**
f **MC**



29 Two forces are applied to a block as shown in the diagram on the right. Calculate the resultant force acting on the block.



30 A police officer travels 3 city blocks south and then 2 blocks east. If each block is 300 m long, find the officer's total displacement as a vector in polar form.

31 A ship is being towed by two tugs. One tug is pulling with a force of 30 000 N in the direction N 23° E, while the other is pulling with a force of 35 000 N in the direction N 46° W. What is the resultant force on the ship?

32 A snooker ball approaching the cushion at an angle of 60° to the cushion has been given spin that makes it bounce off at an angle of 45°. If the ball also changes speed from 0.3 m/s to 0.2 m/s, find its change of velocity.

Practice quiz

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